

Davenport–Hasse’s Theorem for Polynomial Gauss Sums over Finite Fields

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Abstract

In this paper, we study the polynomial Gauss sums over finite fields, and present an analogue of Davenport–Hasse’s theorem for the entire polynomial Gauss sums, which is a generalization of the previous result obtained by Hayes.

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1 Introduction

Let \mathbb{F}_q be a finite field with $q = p^l$ elements, where p is a prime number. Let \mathbb{F}_{q^n} be a finite extension of \mathbb{F}_q of degree n , and σ be the Frobenius on \mathbb{F}_{q^n} , given by $\sigma(a) = a^q$ for any element a in \mathbb{F}_{q^n} . We have $\sigma^n = 1$, and σ generates the Galois group of $\mathbb{F}_{q^n}/\mathbb{F}_q$. The relative trace $\text{tr}(a)$ and the norm $N(a)$ of an element a in \mathbb{F}_{q^n} are defined by

$$\text{tr}(a) = \sum_{i=1}^n \sigma^i(a), \quad N(a) = \prod_{i=1}^n \sigma^i(a) \quad (1.1)$$

respectively. Let ψ be a (complex-valued) character of the additive group of \mathbb{F}_q , and χ a character of the multiplicative group \mathbb{F}_q^* of \mathbb{F}_q . The Gauss sums of \mathbb{F}_q are defined by

$$\tau(\chi, \psi) = \sum_{a \in \mathbb{F}_q^*} \chi(a) \psi(a). \quad (1.2)$$

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If we set for every a in \mathbb{F}_{q^n} that

$$\psi^{(n)}(a) = \psi(\text{tr}(a)), \quad \chi^{(n)}(a) = \chi(N(a)), \quad (1.3)$$

then the function $\psi^{(n)}$ will be a character of additive group, and function $\chi^{(n)}$ a character of the multiplicative group of \mathbb{F}_{q^n} . In particular, if $a \in \mathbb{F}_q$, then we have

$$\psi^{(n)}(a) = \psi^n(a), \quad \chi^{(n)}(a) = \chi^n(a). \quad (1.4)$$

These characters $\psi^{(n)}$ and $\chi^{(n)}$ define a generalized Gauss sums $\tau(\chi^{(n)}, \psi^{(n)})$ on \mathbb{F}_{q^n} . Davenport and Hasse in [4] proved the following remarkable theorem (also see [7] and [9]) that

Theorem 1.1 (Davenport–Hasse) *If both of χ and ψ are not principal, then*

$$-\tau(\chi^{(n)}, \psi^{(n)}) = (-\tau(\chi, \psi))^n. \quad (1.5)$$

To generalize this famous theorem to the polynomial Gauss sums, let $\mathbb{F}_q[x]$ and $\mathbb{F}_{q^n}[x]$ be the polynomial rings over \mathbb{F}_q and \mathbb{F}_{q^n} respectively. The Frobinus σ of \mathbb{F}_{q^n} may be extended to $\mathbb{F}_{q^n}[x]$ in the following way: If $A = a_mx^m + \cdots + a_1x + a_0 \in \mathbb{F}_{q^n}[x]$, then we define

$$\sigma(A) = \sigma(a_m)x^m + \cdots + \sigma(a_1)x + \sigma(a_0), \quad (1.6)$$

which clearly is a $\mathbb{F}_q[x]$ -automorphism of $\mathbb{F}_{q^n}[x]$. The relative trace and norm may be extended to $\mathbb{F}_{q^n}[x]$ by

$$\text{tr}(A) = \sum_{i=1}^n \sigma^i(A), \quad N(A) = \prod_{i=1}^n \sigma^i(A). \quad (1.7)$$

Then $\text{tr}(A)$ is an additive and $N(A)$ a multiplicative function from $\mathbb{F}_{q^n}[x]$ to $\mathbb{F}_q[x]$.

Let H be a fixed but arbitrary polynomial in $\mathbb{F}_q[x]$ with degree m , ψ be a (complex-valued) character of additive group of the residue class ring $\mathbb{F}_q[x]/\langle H \rangle$. We may understand that ψ is a complex-valued function defined on $\mathbb{F}_q[x]$ such that

$$\psi(A + B) = \psi(A) \cdot \psi(B), \text{ and } \psi(A) = \psi(B), \text{ if } A \equiv B \pmod{H} \quad (1.8)$$

for any two polynomials A and B in $\mathbb{F}_q[x]$. According to Hayes [6], we call ψ an additive character modulo H on $\mathbb{F}_q[x]$. For example, ψ_0 is the principal additive character modulo H , where $\psi_0(A) = 1$ for all the polynomials A in $\mathbb{F}_q[x]$.

Let χ be a (complex-valued) character of the multiplicative group of the reduced residue of $\mathbb{F}_q[x]/\langle H \rangle$, χ may be also understood as a complex-valued function of $\mathbb{F}_q[x]$, such that $\chi(A) = 0$ if $(A, H) > 1$ and

$$\chi(AB) = \chi(A) \cdot \chi(B), \text{ and } \chi(A) = \chi(B), \text{ if } A \equiv B \pmod{H}. \quad (1.9)$$

We also call χ a multiplicative character modulo H on $\mathbb{F}_q[x]$. Especially, χ_0 is the principal multiplicative character modulo H , where $\chi_0(A) = 1$ for all of A in $\mathbb{F}_q[x]$ with $(A, H) = 1$.

With the above notations, we define a polynomial Gauss sum $G(\chi, \psi)$ modulo H on $\mathbb{F}_q[x]$ as follows

$$G(\chi, \psi) = \sum_{D \bmod H} \chi(D)\psi(D), \quad (1.10)$$

where the summation extending over a complete residue system of modulo H in $\mathbb{F}_q[x]$.

For a polynomial H in $\mathbb{F}_q[x]$ and, therefore, also a polynomial in $\mathbb{F}_{q^n}[x]$, to define a Gauss sum modulo H on $\mathbb{F}_{q^n}[x]$, for any $A \in \mathbb{F}_{q^n}[x]$, we set $\psi^{(n)}(A)$ and $\chi^{(n)}(A)$ by

$$\psi^{(n)}(A) = \psi(\text{tr}(A)), \quad \chi^{(n)}(A) = \chi(N(A)).$$

It is easy to verify that $\psi^{(n)}$ is an additive and $\chi^{(n)}$ a multiplicative character modulo H on $\mathbb{F}_{q^n}[x]$, thus we may use these characters to define a polynomial Gauss sum $G(\chi^{(n)}, \psi^{(n)})$ modulo H on $\mathbb{F}_{q^n}[x]$.

The most interesting question is that, is there an analogue of Davenport–Hasse’s theorem for the polynomial Gauss sums? Hayes [6, Theorem 2.2] shows that an analogue of Davenport–Hasse’s theorem for a special additive character $\psi = E$, which character was essentially introduced by Carlitz (see [3]). To state Hayes’ result, for any polynomial A in $\mathbb{F}_q[x]$, let

$$A \equiv a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \pmod{H},$$

where $m = \deg(H)$. We set an additive function modulo H on $\mathbb{F}_q[x]$ by $t(A) = a_{m-1}$. By the definition, we have immediately that for any $A, B \in \mathbb{F}_q[x]$, $t(A+B) = t(A) + t(B)$, and $t(A) = t(B)$ whenever $A \equiv B \pmod{H}$, in particular, $t(A) = 0$ if $H|A$. To generalize that t -function modulo H on $\mathbb{F}_q[x]$, for a given polynomial G in $\mathbb{F}_q[x]$, let

$$t_G(A) = t(GA). \quad (1.11)$$

Clearly, t_G is an additive function modulo H on $\mathbb{F}_q[x]$, that is

$$t_G(A+B) = t_G(A) + t_G(B), \text{ and } t_G(A) = t_G(B), \text{ if } A \equiv B \pmod{H}. \quad (1.12)$$

Now, let λ be a fixed non-principal additive character on \mathbb{F}_q , for example, $\lambda(a) = e(\frac{2\pi i \text{tr}(a)}{p})$, for a in \mathbb{F}_q , we define the function $E_\lambda(G, H)$ on $\mathbb{F}_q[x]$ by

$$E_\lambda(G, H)(A) = \lambda(t_G(A)). \quad (1.13)$$

It is easily seen that $E_\lambda(G, H)$ is an additive character modulo H on $\mathbb{F}_q[x]$. If we set $G = 1$, a constant polynomial, and let $E = E_\lambda = E_\lambda(1, H)$, then Hayes [6] shows that an analogue of Davenport–Hasse’s theorem for polynomial Gauss sums in the special case of $\psi = E$.

Theorem 1.2 (Hayes) *If H is a polynomial in $\mathbb{F}_q[x]$ with $\deg(H) = m$, then for any multiplicative character χ of $\mathbb{F}_q[x]$, we have*

$$(-1)^m G(\chi^{(n)}, E^{(n)}) = ((-1)^m G(\chi, E))^n. \quad (1.14)$$

The main purpose of this paper is to generalize the above theorem to all of polynomial Gauss sums, we present a completely analogue of Davenport–Hasse’s theorem in polynomial case.

To state our result, first we note that the character $E_\lambda(G, H)$ given by (1.13) are all of additive characters ψ modulo H on $\mathbb{F}_q[x]$. In other words, for any additive character ψ modulo H on $\mathbb{F}_q[x]$, there exists a unique polynomial G in $\mathbb{F}_q[x]$, such that $\deg(G) < \deg(H)$, and $\psi = E_\lambda(G, H)$ (see Lemma 2.1 below). We write $\psi = \psi_G = E_\lambda(G, H)$, and call G the associated polynomial to ψ , then we have

Theorem 1.3 *If H is a polynomial in $\mathbb{F}_q[x]$ of degree m , χ and ψ are multiplicative and additive characters, not both are principal, then we have*

$$(-1)^{m-m_1} \frac{\phi^{(n)}(N)}{\phi^{(n)}(H)} G(\chi^{(n)}, \psi^{(n)}) = \left((-1)^{m-m_1} \frac{\phi(N)}{\phi(H)} G(\chi, \psi) \right)^n, \quad (1.15)$$

where $N = \frac{H}{(G, H)}$, and G is the associated polynomial to ψ ($\psi = \psi_G$), $m_1 = \deg(G, H)$, $\phi(H)$ is the Euler function on $\mathbb{F}_q[x]$, and $\phi^{(n)}(H)$ is the function on $\mathbb{F}_{q^n}[x]$. In particular, if $H = P^k$, a power of an irreducible P , then we have

$$(-1)^{m-m_1} G(\chi^{(n)}, \psi^{(n)}) = ((-1)^{m-m_1} G(\chi, \psi))^n. \quad (1.16)$$

If $G = 1$ is a constant polynomial, then equality (1.15) becomes Hayes’ result (Theorem 1.2). If $(G, H) = 1$, we also have

$$(-1)^m G(\chi^{(n)}, \psi^{(n)}) = ((-1)^m G(\chi, \psi))^n. \quad (1.17)$$

The equality above essentially belongs to Hayes [6].

As we have known, Davenport–Hasse’s theorem plays an important role for the rationality of the Zeta function associated to a hypersurface, we wish the result presented here are helpful for the congruent Zeta function. Finally, we mention a result given by Thakur that an analogue of Davenport–Hasse’s theorem for Gauss sums taking values in function fields of one variable over a finite field holds, see Thakur [10].

Throughout this paper, by positive polynomial means the polynomial of the leading coefficients is unit in \mathbb{F}_q , the capital letters A, B, C, \dots denote polynomials in $\mathbb{F}_q[x]$, or $\mathbb{F}_{q^n}[x]$, and a, b, c, \dots denote the elements in \mathbb{F}_q or $\mathbb{F}_{q^n}[x]$. The absolute value function $|H| = q^m$, where $m = \deg(H)$, and $|H|_n = q^{nm}$ on $\mathbb{F}_{q^n}[x]$, which are the numbers of a complete residue class module H on $\mathbb{F}_q[x]$ and $\mathbb{F}_{q^n}[x]$.

2 Properties of character $E_\lambda(G, H)$

In this section, we first determine the construction of the additive character group modulo H on $\mathbb{F}_q[x]$ by using $E_\lambda(G, H)$.

Lemma 2.1 *For any ψ , an additive character modulo H on $\mathbb{F}_q[x]$, there exists a unique polynomial G in $\mathbb{F}_q[x]$, such that $\deg(G) < \deg(H)$, and $\psi = E_\lambda(G, H)$.*

Proof For the convenient sake, we write

$$\psi_G = E_\lambda(G, H). \quad (2.1)$$

By (1.13), we have $\psi_{G_1} = \psi_{G_2}$, whenever $G_1 \equiv G_2 \pmod{H}$, so we may set G in a complete residue class modulo H in $\mathbb{F}_q[x]$, and then $\deg(G) < \deg(H)$. It is easy to show that by (1.13), for any G_1, G_2 in $\mathbb{F}_q[x]$ that

$$\psi_{G_1+G_2} = \psi_{G_1} \cdot \psi_{G_2}, \text{ and } \bar{\psi}_G = \psi_{-G}. \quad (2.2)$$

Since $\psi_G = \psi_0$ the principal character modulo H on $\mathbb{F}_q[x]$, if $G = 0$, or $H|G$. Conversely, we have $\psi_G = \psi_0$, if and only if $H|G$. Since λ is a non-principal character on \mathbb{F}_q by assumption, then there is an element a in \mathbb{F}_q , so that $\lambda(a) \neq 1$. Now if $H \nmid G$, we may let

$$R = (G, H) = a_k x^k + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x], \quad (2.3)$$

where $0 \leq k \leq m-1$, and $a_k \neq 0$. It follow that

$$a \cdot a_k^{-1} x^{m-1-k} R = ax^{m-1} + \cdots.$$

We note that the congruent equation in variable T that

$$GT \equiv a \cdot a_k^{-1} x^{m-1-k} R \pmod{H}, \quad (2.4)$$

is solvable in $\mathbb{F}_q[x]$, therefore, there exists a polynomial A in $\mathbb{F}_q[x]$, such that

$$GA \equiv a \cdot a_k^{-1} x^{m-1-k} R \pmod{H}, \quad (2.5)$$

and we see that $t_G(A) = a$ by the definition of (1.11), and $\psi_G(A) = \lambda(t_G(A)) = \lambda(a) \neq 1$, and $\psi_G \neq \psi_0$. By (2.2), we have immediately that

$$\psi_{G_1} \neq \psi_{G_2}, \text{ if } G_1 \not\equiv G_2 \pmod{H}. \quad (2.6)$$

Since if $\psi_{G_1} = \psi_{G_2}$, then $\psi_{G_1-G_2} = \psi_0$, and $G_1 \equiv G_2 \pmod{H}$. This shows that ψ_G are different from each other when G running through a complete residue system of modulo H , hence there are exactly $|H| = q^m$ different characters ψ_G , but the number of additive characters modulo H on $\mathbb{F}_q[x]$ exactly is q^m , thus every additive character ψ is just the form of ψ_G . We complete the proof of Lemma 2.1. \square

The next lemma is not new, one may find in Carlitz [3] (see [3](2.4), (2.5), and (2.6)), but we give a more explicit expression.

Lemma 2.2 *If A is a positive polynomial in $\mathbb{F}_q[x]$, then we have*

$$E_\lambda(GA, HA) = E_\lambda(G, H). \quad (2.7)$$

Proof For any $B \in \mathbb{F}_q[x]$, let

$$GB \equiv a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \pmod{H}.$$

Then

$$AGB \equiv A(a_{m-1}x^{m-1} + \cdots + a_1x + a_0) \pmod{AH}.$$

Because of A a positive polynomial, we see that the function t_{GA} modulo HA just is the function t_G modulo H . Since that $t_{GA}(B) = a_{m-1}$ modulo HA , which is just $t_G(B)$ modulo H . It follows that

$$E_\lambda(GH, HA)(B) = \lambda(t_G(B)) = E_\lambda(G, H)(B), \quad (2.8)$$

and we have the lemma at once. \square

Lemma 2.3 *For any A in $\mathbb{F}_q[x]$, we have*

$$\sum_{G \bmod H} \psi_G(A) = \begin{cases} |H|, & \text{if } H \mid A \\ 0, & \text{otherwise} \end{cases},$$

where the summation extending over a complete residue system modulo H .

Proof By Lemma 2.1, it just is the orthogonal property of characters. We have the lemma immediately. \square

3 The separable polynomial Gauss sums

The theory for conductors of modulo a polynomial parallels the theory for conductors of Dirichlet characters defined on the integers (see, e.g., [1, pp. 165–172]), but for proving our theorem, we still state and prove a few basic results. First by Lemma 2.1, all of the Gauss sums on $\mathbb{F}_q[x]$ may be written by

$$G(\chi, \psi) = G(\chi, \psi_G) = \sum_{D \bmod H} \chi(D) \psi_G(D), \quad (3.1)$$

where $G \in \mathbb{F}_q[x]$ is the polynomial associated to ψ . If $(G, H) = 1$, it is easy to verify that

$$G(\chi, \psi_G) = \bar{\chi}(G) G(\chi, \psi_1). \quad (3.2)$$

Definition 3.1 A Gauss sum $G(\chi, \psi_G)$ is said to be separable if $G(\chi, \psi_G) = \bar{\chi}(G) G(\chi, \psi_1)$.

By (3.2), if $(G, H) = 1$, then $G(\chi, \psi_G)$ is separable. For the case of $(G, H) > 1$, then $G(\chi, \psi_G)$ is separable if and only if $G(\chi, \psi_G) = 0$. The following lemma gives an important consequence of separability.

Lemma 3.1 *If $G(\chi, \psi_G)$ is separable for every G in $\mathbb{F}_q[x]$, then*

$$|G(\chi, \psi_1)|^2 = |H| = q^m. \quad (3.3)$$

Proof

$$\begin{aligned} |G(\chi, \psi_1)|^2 &= G(\chi, \psi_1) \overline{G(\chi, \psi_1)} \\ &= G(\chi, \psi_1) \sum_{D \bmod H} \bar{\chi}(D) \psi_1(-D) \\ &= \sum_{D \bmod H} G(\chi, \psi_D) \psi_D(-1) \\ &= \sum_{A \bmod H} \chi(A) \sum_{D \bmod H} \psi_D(A-1). \end{aligned} \quad (3.4)$$

The inner sum in above is zero by Lemma 2.3, if $A \not\equiv 1 \pmod{H}$, so we have Lemma 3.1. \square

Lemma 3.2 *If $G(\chi, \psi_G) \neq 0$ for some G in $\mathbb{F}_q[x]$ with $(G, H) > 1$, then there exists a polynomial N in $\mathbb{F}_q[x]$, such that $N|H$, $\deg(N) < \deg(H)$, and*

$$\chi(A) = 1, \text{ whenever } (A, H) = 1, \text{ and } A \equiv 1 \pmod{N}. \quad (3.5)$$

Proof For given G , and $G(\chi, \psi_G) \neq 0$, $(G, H) > 1$. Let $N = H \cdot (G, H)^{-1}$, thus $N|H$, and $\deg(N) < \deg(H)$. If $(A, H) = 1$, then

$$\begin{aligned} G(\chi, \psi_G) &= \sum_{D \bmod H} \chi(AD) \psi_G(AD) \\ &= \chi(A) \sum_{D \bmod H} \chi(D) \psi_G(AD). \end{aligned} \quad (3.6)$$

If $A \equiv 1 \pmod{N}$, we write

$$A = 1 + BN, \text{ for some } B \in \mathbb{F}_q[x],$$

and then

$$AG = G + BNG = G + BHG(H, G)^{-1}.$$

So we have $AG \equiv G \pmod{H}$, and $\psi_G(AD) = \psi_{GA}(D) = \psi_G(D)$. Therefore, equation (3.6) becomes that

$$G(\chi, \psi_G) = \chi(A) \sum_{D \bmod H} \chi(D) \psi_G(D), \quad (3.7)$$

and we have $\chi(A) = 1$ because of $G(\chi, \psi_G) \neq 0$. We complete the proof of Lemma 3.2. \square

Definition 3.2 A polynomial N in $\mathbb{F}_q[x]$ is called an induced modulu of χ if $N|H$, and for $(A, H) = 1$

$$\chi(A) = 1, \text{ whenever } A \equiv 1 \pmod{N}. \quad (3.8)$$

By the definition, we see that H itself is an induced modulu of any χ . Moreover, as a direct consequence of Lemma 3.2, we also have

Corollary 3.3 If $(G, H) > 1$, and $G(\chi, \psi_G) \neq 0$, then $N = H(G, H)^{-1}$ is an induced modulu of χ .

Lemma 3.4 Let $N|H$, then N is an induced modulu of χ if and only if for any A, B in $\mathbb{F}_q[x]$, and $(AB, H) = 1$, we have

$$\chi(A) = \chi(B), \text{ whenever } A \equiv B \pmod{N}. \quad (3.9)$$

Proof If (3.9) holds, let $B = 1$, then N is an induced modulu of χ . Conversely, if N is an induced modulu of χ , suppose $(A, H) = (B, H) = 1$, $A \equiv B \pmod{N}$, and let $B \cdot B^{-1} \equiv 1 \pmod{H}$. Then $BB^{-1} \equiv 1 \pmod{N}$, and $AB^{-1} \equiv 1 \pmod{N}$. Thus $\chi(AB^{-1}) = 1$, and $\chi(A) = \chi(B)$, the lemma follows. \square

Lemma 3.5 If $N|H$, and N is an induced modulu of χ , then χ can be expressed as a product

$$\chi = \chi_0 \delta, \quad (3.10)$$

where χ_0 is the principal multiplicative modulo H , and δ is a multiplicative character modulo N .

Proof If $\chi = \chi_0 \delta$, trivially, N is an induced modulu of χ . Conversely, if N is an induced modulu, we may determine a character δ modulo N by setting $\delta(A) = 0$, if $(A, N) > 1$. If $(A, N) = 1$, one may find a polynomial B in $\mathbb{F}_q[x]$, so that

$$(B, H) = 1, \text{ and } B \equiv A \pmod{N}. \quad (3.11)$$

Since the arithmetic progress $\{A + RN | R \in \mathbb{F}_q[x]\}$ contains infinitely many irreducibles (see Artin [2], or [8, Theorem 4.7]), so we may choose one that does not divide H and call this B , which is unique modulo N clearly. Now we define $\delta(A) = \chi(B)$. The number $\delta(A)$ is well-defined because χ takes equal values at polynomials which are congruent modulo N and relatively prime to H . By this determination, we can easily verify that δ is, indeed, a character modulo N , and (3.11) holds. This is the proof of Lemma 3.5. \square

Definition 3.3 An induced modulu N of χ is called the conductor of χ , if N is positive, and an induced modulu of χ , and the degree of N is least among all of induced modulus of χ . We denote by C_χ the conductor of χ . If $C_\chi = H$, then we call χ is a primitive character.

As a consequence of Lemma 3.5, we have

Corollary 3.6 *If C_χ is the conductor of χ , then χ can be expressed as a product $\chi = \chi_0\delta$, where δ is a primitive character modulo C_χ .*

The following lemma is well-known that

Lemma 3.7 *The conductor of χ divides every induced modulu of χ .*

Proof See Hayes [6, Theorem 4.2]. \square

We have an alternate description of primitive character as the case of Dirichlet characters.

Lemma 3.8 *Let χ be a character modulo H , then χ is primitive, if and only if the Gauss sums $G(\chi, \psi_G)$ is separable for every polynomial G .*

Proof If χ is primitive, then $G(\chi, \psi_G)$ is separable by Lemma 3.2, so we only prove the converse. It suffices to prove that if χ is not primitive, then there exists some G with $(G, H) > 1$, and $G(\chi, \psi_G) \neq 0$. If χ is not primitive, let C_χ be the conductor of χ , $N = \frac{H}{C_\chi}$, then $(N, H) > 1$ by $\deg(C_\chi) < \deg(H)$, moreover $G(\chi, \psi_N) \neq 0$. Since $\chi = \chi_0\delta$, where δ is primitive character modulo C_χ . By Lemma 2.2, we have

$$\begin{aligned}
G(\chi, \psi_N) &= \sum_{D \bmod H} \chi_0(D) \delta(D) \psi_N(D) \\
&= \sum_{\substack{D \bmod H, \\ (D, H)=1}} \delta(D) E_\lambda(N, H)(D) \\
&= \sum_{\substack{D \bmod H, \\ (D, H)=1}} \delta(D) E_\lambda(1, C_\chi)(D) \\
&= \frac{\phi(H)}{\phi(C_\chi)} \sum_{D \bmod C_\chi} \delta(D) \psi_1(D) \\
&= \frac{\phi(H)}{\phi(C_\chi)} G(\delta, \psi_1),
\end{aligned} \tag{3.12}$$

where $G(\delta, \psi_1)$ is a Gauss sum modulo C_χ . By Lemma 3.1

$$|G(\delta, \psi_1)|^2 = |C_\chi|. \tag{3.13}$$

So we have $G(\chi, \psi_N) \neq 0$, and the lemma follows. \square

4 Separable Gauss sums on $\mathbb{F}_{q^n}[x]$

For a polynomial H in $\mathbb{F}_q[x]$, and also a polynomial in $\mathbb{F}_{q^n}[x]$. Let $\psi_G^{(n)}$ be an additive character, $\chi^{(n)}$ a multiplicative character modulo H on $\mathbb{F}_{q^n}[x]$ given by (1.10). We recall that $\psi_G = E_\lambda(G, H)$ in $\mathbb{F}_q[x]$, it still holds in $\mathbb{F}_{q^n}[x]$, namely

$$\psi_G^{(n)} = E_\lambda^{(n)}(G, H), \text{ if } G \in \mathbb{F}_q[x]. \quad (4.1)$$

Since for any polynomial A in $\mathbb{F}_{q^n}[x]$

$$E_\lambda^{(n)}(G, H)(A) = E_\lambda(G, H)(\text{tr}(A)) = \lambda(t_G(\text{tr}(A))) = \psi_G^{(n)}(A).$$

The last equality is because of $\text{tr}(G \cdot A) = G \text{tr}(A)$ for $G \in \mathbb{F}_q[x]$. So we have if $G \in \mathbb{F}_q[x]$ then

$$\psi_G^{(n)}(A) = \psi_1^n(GA). \quad (4.2)$$

Now, Lemma 2.2 becomes that

Lemma 4.1 *If $A \in \mathbb{F}_q[x]$ is a positive polynomial, and $G \in \mathbb{F}_q[x]$, then*

$$E_\lambda^{(n)}(GA, HA) = E_\lambda^{(n)}(G, H). \quad (4.3)$$

Proof For any $B \in \mathbb{F}_{q^n}[x]$, let

$$G \text{tr}(B) \equiv a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \pmod{H}.$$

Then

$$AG \text{tr}(B) \equiv A(a_{m-1}x^{m-1} + \cdots + a_1x + a_0) \pmod{AH}.$$

By definition of $E_\lambda(G, H)$ and $E_\lambda^{(n)}(G, H)$, it follows that

$$E_\lambda^{(n)}(GA, HA)(B) = E_\lambda(GA, HA)(\text{tr}(B)) = \lambda(t_{GA}(\text{tr}(B))) = \lambda(a_{m-1}),$$

and

$$E_\lambda^{(n)}(G, H)(B) = E_\lambda(G, H)(\text{tr}(B)) = \lambda(t_G(\text{tr}(B))) = \lambda(a_{m-1}).$$

This lemma follows at once. \square

Lemma 4.2 *If $N \in \mathbb{F}_q[x]$ is an induced modulu of χ , then N is also an induced modulu of $\chi^{(n)}$ on $\mathbb{F}_{q^n}[x]$.*

Proof Suppose $A \in \mathbb{F}_{q^n}[x]$, $(A, H) = 1$, and $A = 1 \pmod{N}$, it is easily seen that if $N(A)$ is the norm that

$$N(A) \equiv 1 \pmod{N}. \quad (4.4)$$

This equality is the Theorem 2.1 of Hayes [6]. Then $\chi^{(n)}(A) = \chi(N(A)) = 1$, and N is an induced modulu of $\chi^{(n)}$ on $\mathbb{F}_{q^n}[x]$. \square

The following lemmas is due to Hayes [6] that

Lemma 4.3 *Let C_χ be the conductor of χ on $\mathbb{F}_q[x]$, and $C_{\chi^{(n)}}$ be the conductor of $\chi^{(n)}$ on $\mathbb{F}_{q^n}[x]$, then we have $C_\chi = C_{\chi^{(n)}}$.*

Proof See Hayes [6, Theorem 4.5]. \square

As a direct corollary of the above lemma and Lemma 3.8, we have

Corollary 4.4 *Let $G(\chi, \psi)$ be the Gauss sums modulo H on $\mathbb{F}_q[x]$, $G(\chi^{(n)}, \psi^{(n)})$ the Gauss sums modulo H on $\mathbb{F}_{q^n}[x]$, then $G(\chi^{(n)}, \psi^{(n)})$ is separable if and only if $G(\chi, \psi)$ is separable.*

Lemma 4.5 *Suppose G and H are in $\mathbb{F}_q[x]$, and $(G, H) > 1$, if $G(\chi, \psi_G) \neq 0$, or $G(\chi^{(n)}, \psi_G^{(n)}) \neq 0$, then $N = \frac{H}{(G, H)}$ is an induced modulu of both χ and $\chi^{(n)}$.*

Proof It is easily seen that if N is an induced modulu of χ , then any multiple of N which divides H , again is an induced modulu of χ . To prove the lemma, first suppose $G(\chi, \psi_G) \neq 0$, because $(G, H) > 1$, then by Lemma 3.2 and Corollary 3.3, N is an induced modulu of χ . By Lemma 4.2, N is also an induced modulu of $\chi^{(n)}$, the lemma holds. If $G(\chi^{(n)}, \psi_G^{(n)}) \neq 0$, then N is an induced modulu of $\chi^{(n)}$. Let $C_{\chi^{(n)}}$ be the conductor of $\chi^{(n)}$, then by Lemma 3.7, we have $C_{\chi^{(n)}} | N$, and then $C_\chi | N$, where C_χ is the conductor of χ . Therefore N is an induced modulu of χ . We complete the proof of Lemma 4.5. \square

5 Proof of Theorem 1.3

We consider two cases to prove this theorem. First, if $(G, H) = 1$, then $(G, H) = 1$ in $\mathbb{F}_{q^n}[x]$. The Gauss sums on $\mathbb{F}_{q^n}[x]$ is that

$$G(\chi^{(n)}, \psi_G^{(n)}) = \bar{\chi}^{(n)}(G)G(\chi^{(n)}, \psi_1^{(n)}), \quad (5.1)$$

and the Gauss sums $G(\chi, \psi_G)$ on $\mathbb{F}_q[x]$ is following

$$G(\chi, \psi_G) = \bar{\chi}(G)G(\chi, \psi_1). \quad (5.2)$$

We note that $\chi^{(n)}(G) = \chi^n(G)$, $\psi_1 = E$, by Theorem 1.2, we have

$$\begin{aligned} (-1)^m G(\chi^{(n)}, \psi_G^{(n)}) &= (-1)^m \bar{\chi}^{(n)}(G)G(\chi^{(n)}, \psi_1^{(n)}) \\ &= ((-1)^m \bar{\chi}(G)G(\chi, \psi_1))^n \\ &= ((-1)^m G(\chi, \psi_G))^n. \end{aligned} \quad (5.3)$$

Because of $(G, H) = 1$, then $m_1 = \deg(G, H) = 0$, and $\phi^{(n)}(H) = \phi^{(n)}(N)$, $\phi(H) = \phi(N)$, (1.15) of Theorem 1.3 holds in the case of $(G, H) = 1$.

Next, we suppose $(G, H) > 1$, and let $H_1 = \frac{H}{(G, H)}$, $G_1 = \frac{G}{(G, H)}$, thus $(G_1, H_1) = 1$. If both of $G(\chi, \psi_G)$ and $G(\chi^{(n)}, \psi_G^{(n)})$ are zero, then (1.15) is trivial. Therefore, we may assume $G(\chi, \psi_G) \neq 0$, or $G(\chi^{(n)}, \psi_G^{(n)}) \neq 0$. By this assumption, then H_1 is an induced modulu of both χ and $\chi^{(n)}$. By Lemma 3.4, we may write

$$\chi = \chi_0 \delta, \text{ and } \chi^{(n)} = \chi_0^{(n)} \delta^{(n)}, \quad (5.4)$$

where δ is a multiplicative character modulo H_1 , and $\delta^{(n)}(A) = \delta(N(A))$ is a multiplicative character modulo H_1 on $\mathbb{F}_{q^n}[x]$. The Gauss sums on $\mathbb{F}_{q^n}[x]$ is that

$$\begin{aligned} G(\chi^{(n)}, \psi_G^{(n)}) &= \sum_{\substack{D \bmod H, \\ D \in \mathbb{F}_{q^n}[x]}} \chi_0^{(n)}(D) \delta^{(n)}(D) \psi_G^{(n)}(D) \\ &= \sum'_{\substack{D \bmod H, \\ D \in \mathbb{F}_{q^n}[x]}} \delta^{(n)}(D) E_\lambda^{(n)}(G, H)(D) \\ &= \sum'_{\substack{D \bmod H, \\ D \in \mathbb{F}_{q^n}[x]}} \delta^{(n)}(D) E_\lambda^{(n)}(G_1, H_1)(D) \\ &= \frac{\phi^{(n)}(H)}{\phi^{(n)}(H_1)} G(\delta^{(n)}, \psi_{G_1}^{(n)}), \end{aligned} \quad (5.5)$$

where summation \sum' means $(D, H) = 1$, and $G(\delta^{(n)}, \psi_{G_1}^{(n)})$ is a Gauss sums modulo H_1 . The Gauss sums $G(\chi, \psi_G)$ modulo H on $\mathbb{F}_q[x]$ is that

$$\begin{aligned} G(\chi, \psi_G) &= \sum'_{D \bmod H} \delta(D) \psi_G(D) \\ &= \sum'_{D \bmod H} \delta(D) E_\lambda(G, H)(D) \\ &= \sum'_{D \bmod H} \delta(D) E_\lambda(G_1, H_1) \\ &= \frac{\phi(H)}{\phi(H_1)} G(\delta, \psi_{G_1}), \end{aligned} \quad (5.6)$$

where $G(\delta, \psi_{G_1})$ is a Gauss sums modulo H_1 . Because of $(G_1, H_1) = 1$, the discussion for first case gives us that

$$(-1)^{m-m_1} G(\delta^{(n)}, \psi_{G_1}^{(n)}) = ((-1)^{m-m_1} G(\delta, \psi_{G_1}))^n, \quad (5.7)$$

where $m - m_1 = \deg(H_1)$, and the equality (1.15) of Theorem 1.3 follows immediately.

To prove (1.16) of Theorem 1.3, if $H|G$, then $\psi_G = \psi_0$ is the principal character modulo H , then both sides of (1.16) are zero, if χ is non-principal, then we may suppose that $H \nmid G$.

Since $H = P^k$, where $k \geq 1$, and P is an irreducible in $\mathbb{F}_q[x]$, it is well-known that P is product of exactly (h, n) irreducibles in $\mathbb{F}_{q^n}[x]$, where $h = \deg(P)$ (see [5, Theorem 2.1], for example). If $H \nmid G$, then $\frac{H}{(G, H)} = N = P^{k_1}$, where $1 \leq k_1 \leq k$, it is easy to verify that

$$\phi^{(n)}(N)(\phi^{(n)}(H))^{-1} = (\phi(N)\phi^{-1}(H))^n. \quad (5.8)$$

So (1.16) follows from (1.15), we complete the proof of Theorem 1.3.

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